On the determination of sea level changes by combining altimetric, tide gauge, satellite gravity and atmospheric observations

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Abstract. The determination, monitoring and understanding of sea level change at various spatial and temporal scales has been the focus of many studies during the past decades. The advent of satellite altimetry and the multitude of unprecedented in accuracy and resolution observations that it offers allowed, in combination with tide gauge data, precise determinations of sea level variations. The realization of the GRACE mission and the forthcoming GOCE data offer new opportunities for the estimation of sea level trends at regional and global scales and the identification of seasonal signals. In such studies, even though the data combination and processing strategies have been carried out carefully with proper control, a point that has been given little attention is error propagation and variance component estimation of the data variance-covariance matrices. The latter two are of significant importance in heterogeneous data combination studies, since on one hand error propagation can provide reliable estimates of the output signal error while variance component estimation allows for a rigorous control of the data covariance matrices and subsequent sound decisions on statistical testing of hypotheses involving least-squares residuals and the estimated deterministic parameters. This work presents some new ideas towards the estimation of sea level change through the combination of altimetric, tide gauge, atmospheric, and GRACE- and GOCE-type observables. The combination scheme is based on a hybrid deterministic and stochastic treatment of the data errors and an estimation of sea level changes through least-squares collocation with considerations for glacial isostatic adjustment and continental water outpour effects. The deterministic model parameters treat datum and geophysical correction model inconsistencies in the data used, while the stochastic part allows for a simultaneous determination of stochastic parameters included in the data in terms of residual signals. Within this mixed adjustment scheme with stochastic parameters, variance component estimation is carried out using the iterative almost unbiased estimator method. The analytical equations for the prediction of the adjusted input and output signals are presented along with possible modifications of the observation equation for the determination of solely steric and atmospheric-driven sea level changes.

Keywords. sea level change, altimetry, satellite gravimetry, collocation, tide gauge.

1 Introduction

Sea level changes at global and regional scales are triggered by a number of factors that take place within system Earth. These natural processes originate from variations in the physical properties of the ocean water and from water mass transport between the Earth's oceans, continents and the atmosphere. The former, refers to variations in the seawater density triggered by salinity and temperature changes and results in so-called steric sea level changes. On the other hand, water mass transport due to changes in the continental reservoirs (river run-off), glacial and ice caps mass variations (melting) and atmospheric water vapor changes (precipitation and evaporation), trigger the so-called non-steric (eustatic) sea level changes (Chen et al. 2005, Chambers et al. 2004, Garcia et al. 2007). A traditional tool to monitor sea level changes at local, national and regional scales has been the deployment of tide-gauge (TG) stations along the coastline and their unification in a common vertical datum in order to refer to the same zero plane. The main disadvantages of such tide-gauge networks were and still are the inability to provide global estimates for the mean sea level change, the operational costs, the cumbersome nature of the measurements and the non-unified accuracy of the final sea level estimates. On the other hand, since tide-gauge stations exist for long periods of time, their record of sea level variations extends to many decades.

The advent of satellite altimetry in the early 80's resulted in the availability of sea surface height measurements with global coverage, homogeneous accuracy and resolution. These observations form an unprecedented database to monitor variations of the sea level at global scales, without the limitations of ground-based observations. Moreover, the original exact repeat missions (ERM) of ERS1 and TOPEX/Poseidon were followed by ERS2/ENVISAT and JASON1/JASON2 respectively, which are set at the exact same orbit as their predecessors (Chelton et al. 2001). In that way, a 20-year record of observations for the sea level is available from altimeters on-board satellites. The only disadvantages in the altimetric measurements for sea level monitoring are that a) their record is not as long as that of the tide-gauges, so that no definite conclusions can be drawn w.r.t. long-term predictions and trends for the sea level variations and b) due to the scattering of the radar pulse close to the coastline and in shallow-water regions their correlation with TG measurements not trivial. It should be pointed out that the TG and altimetric observations record both components, steric and non-steric, of the sea level change. Therefore if one would like to determine only the steric component then oceanographic observations about the salinity and temperature need to be employed. On the other hand, eustatic sea level changes imply the use of some models for water mass changes within the ocean, continents and atmosphere system.

This situation arrived at a landmark in March 2002, when the Gravity Recovery and Climate Experiment (GRACE) mission was launched. The twin GRACE satellites operating in SST-ll (low-low satellite to satellite tracking) mode, managed to provide invaluable observations of the spatio-temporal variations of the Earth's gravity field. These are utilized through the 10-day and monthly gravity field models generated from GRACE data, so that their differences with a mean (static) gravity field yield gravity variations. The variations are expressed as lateral mass changes under the assumption that they are caused by redistributions of the continental, oceanic and atmospheric water mass (Garcia et al. 2006). Based on its instrumentation, orbit and measurement precision, GRACE quantifies vertical integrated water mass changes with an accuracy of a few mm for spatial scales of ~400 km. Water mass changes are expressed as geoid height anomalies with respect to a static field, by evaluating time-dependent ($\delta C_{nm}(t)$ and $\delta S_{nm}(t)$) spherical harmonic coefficients (Lombard et al. 2007, Swenson and Wahr 2002). The latter, result as differences between the temporal and mean expansions of the Earth's gravity field in spherical harmonics coefficients.

From the previous analysis it becomes evident that new prospects, as far as sea level change monitoring is concerned, are offered. A combination scheme that utilizes satellite altimetry, GRACE and tide gauge data would in fact manage to lead to the optimal estimation of both steric and non-steric sea level anomalies (SLA) at variable spatial and time scales. If these observations are coupled with the forthcoming GOCE measurements of the second order derivatives of the Earth's potential, which will result to more accurate estimates of the static geoid, then a heterogeneous data combination methodology can be outlined in order to estimate both sea level changes as well as the time-varying and quasi-stationary sea surface topography. In modern day geodetic research one of the most versatile, among others (Sansò and Sideris 1997, Tziavos et al. 1998), and rigorous combination methods is that based on least-squares collocation (Moritz 1980). This study focuses on the presentation of a combination scheme based on a hybrid deterministic and stochastic treatment of the data errors and an estimation of sea level changes through least-squares collocation (LSC). The deterministic model parameters treat datum and geophysical correction model inconsistencies in the data used (satellite altimetry and GRACE observations), while the stochastic part allows for a simultaneous determination of stochastic parameters included in the data in terms of residual signals. Analytical equations for the prediction of the adjusted input and output signals are presented along with reliable estimates of the output signal error covariance matrices. Finally, within this mixed adjustment scheme with stochastic parameters, a note on variance component estimation is carried out either using minimum norm quadric unbiased estimation or iterative almost unbiased estimator methods.

2 Data and observation equations

The analysis presented herein is based on the assumption that satellite altimetry, GRACE, GOCE and tide-gauge observations are available. Furthermore, we assume that all time-varying observations refer to the same epoch, thus we neglect the variable of time t in the observation equations. For multi-epoch data, e.g., $t_i \left\{i = 1,2,...,n\right\}$, n-times more observations equations would be available.

2.1 Satellite altimetry observations

Altimeters on-board satellites practically measure the instantaneous height of the sea surface from a reference ellipsoid. The observation equation for the altimetric measurements can be written as:

$$h^{SSH} = h_{alt}^{SLA} + h^{MSS} + v^{alt}, \qquad (1)$$

where h^{SSH} are the altimetric sea surface heights (SSHs), h_{alt}^{SLA} denotes the sea level anomaly (SLA) as observed by the altimeter, h^{MSS} denotes the mean sea surface (MSS) and v^{alt} are the errors of the observations, which are estimated to be ~4 mm (Chelton et al. 2001). Note that h_{alt}^{SLA}, contrary to GRACE measurements, contains both the steric an the non-steric components of the sea level variations. Some considerations for the altimetric measurements are a) all geophysical and instrumental corrections need to be applied, b) they also need to be corrected for the inverse barometric (IB) effect, so that the total ocean mass signal will be accounted for (Garcia et al. 2007, Vergos et al. 2005), and c) the contribution of the glacial isostatic adjustment (GIA), which reduces the global mean sea level by ~-0.3 mm/y needs to be accounted for in order to explain the altimeter sea level rise due to climate factors (Lombard et al. 2007). Note that the IB correction should be exercised with care, since GRACE data observe the real water mass signal so in order to be consistent with that altimetric observations should not be IB-corrected in that case. On the other hand if we are interested on the total ocean mass signal then the IB correction should be applied to altimetry data and restored to GRACE observations. In Eq. (1) h_{alt}^{SLA} contains both the steric and non-steric SLA, so it can be written as $h_{alt}^{SLA} = h_{steric}^{alt^{SLA}} + h_{non\text{-}steric}^{alt^{SLA}}$ where the steric effect equals to that of the time-varying sea surface topography (SST) $\delta\zeta$. Knowing that the MSS height can be expressed in terms of the geoid height N and the quasi-stationary SST ζ^c we can re-write Eq. (1) as:

$$h^{SSH} = h_{alt}^{SLA} + N + \zeta^c + v^{alt} . \quad (2)$$

If multi-mission satellite altimetry data are to be used, bias a^{alt} and tilt b^{alt} parameters can be introduced in Eq. (2) to account for orbital errors, multi-mission MSL deviations, remaining tidal-effects, datum inconsistencies, etc. Therefore the final observation equation for altimetric data can be formed

$$h^{SSH} = h_{alt}^{SLA} + N + \zeta^c + a^{alt} + b^{alt}\delta t + v^{alt} . \quad (3)$$

2.2 Tide-gauge and GPS observations

For the incorporation of tide-gauge observations we assume that a local TG is available so that a locally determined MSS H_{TG}^{MSS} from its series of measurements is available. Moreover, it is equipped with a GPS receiver so that its ellipsoidal height h^{TG} can be determined. Then we can write the initial observation equation for the TG measurements as:

$$h^{TG} = H_{TG}^{MSS} + N + \zeta^c + a^{TG} + v^{TG} , \quad (4)$$

where v^{TG} are the errors of the observations and a^{TG} is a bias parameter to account for the deviation of the local MSL, determined by the TG station, compared to the global one determined by altimetry. Note that this bias parameter can be also regarded as the deviation between a so-called local geoid (local equipotential surface W_o determined by the TG measurements) and a global geoid. It is acknowledged that the incorporation of GPS observations to TG data limits their record, but it is required in order to relate them to geoid heights. The term H_{TG}^{MSS} can be further decomposed in order to include the instantaneous observations of the TG station H_{TG}^{ISL} and the SLA h^{SLA}, since $H_{TG}^{MSS} = H_{TG}^{ISL} + h^{SLA}$. Note that h^{SLA} is no more the local SLA observed by the TG, but the so-called global one since the bias term a^{TG} has been included in the observation equation. If we further decompose the errors of the observations in those due to the GPS measurements and the ISL and SLA measurements of the TG we arrive at the final observation equation for the TG records:

$$h^{TG} = H_{TG}^{ISL} + h^{SLA} + N + \zeta^c + a^{TG} + v^{h_{TG}} + v^{H_{TG}^{ISL}} + v^{h^{SLA}} . \quad (5)$$

2.3 GOCE observations

GOCE, was launched in March 17, 2009 and through its satellite gradiometry measurement principle will provide observations of the disturbing potential and its second order derivatives. It is foreseen that the geopotential models that will result from GOCE, in the form of spherical harmonic expansions of the Earth's potential, will have a cumulative geoid accuracy of 1 cm to degree and order 200. For GOCE observations to be included in the combination scheme, we assume that a) data for the disturbing potential T and its second order derivatives T_{rr} are available and b) these observations are processed ones, i.e., they represent the static gravity field as sampled by GOCE. The observation equations can then be written as:

$$T = T^{GOCE} + v^{T^{GOCE}} , \quad (6)$$

$$T_{rr} = T_{rr}^{GOCE} + v^{T_{rr}^{GOCE}} . \quad (7)$$

In Eqs. (6) and (7) v^T and $v^{T_{rr}}$ are prediction errors of T and T_{rr} with given covariance and cross-covariance structure.

2.4 GRACE observations

As it has already been mentioned, GRACE data are available in terms of spherical harmonics expansion of the geoid as static 10-day and monthly fields. Since we are interested in mass variations we will utilize the differences between a variable and the static model in order to have available at hand geoid height (actually potential) changes. These can be translated into water thickness equivalent, thus allowing the determination of sea level changes. It should be once again stressed that GRACE data over the oceanic domain *observe* only the variations of the sea level due to ocean mass change and not the steric part of it. Lets assume that we take the difference between a GRACE monthly solution w.r.t. to a mean field then the geoid height variation $N(t)$ can be expressed as

$$\delta N(t) = R\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{\bar{P}_{nm}\left(\cos\theta\right)\left[\delta\bar{C}_{nm}\left(t\right)\cos m\lambda + \delta\bar{S}_{nm}\left(t\right)\sin m\lambda\right]\right\}, \qquad (8)$$

where *δ{.}* denotes difference and all other terms in Eq. (8) are known and need not be defined (Heiskanen and Moritz 1967). Then surface mass variations can be determined (Garcia et al. 2006)

$$\Delta\sigma = \frac{R\rho_E}{3}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{\left(\frac{2n+1}{1+\kappa_n}\right)\bar{P}_{nm}\left(\cos\theta\right)\left[\delta\bar{C}_{nm}\cos m\lambda + \delta\bar{S}_{nm}\sin m\lambda\right]\right\}, \qquad (9)$$

where ρ_E is a mean density of solid Earth and κ_n denotes load love numbers. Finally, utilizing the mass variations $\Delta\sigma$ GRACE inferred SLA can be determined:

$$h^{SLA}_{GRACE} = \frac{\Delta\sigma}{1.029}[\mathrm{mm}], \qquad (10)$$

If we combine the GRACE SLA observations with the corresponding ones from altimetry, we can write the final observation equation for GRACE data

$$h^{SLA}_{GRACE} = h^{SLA}_{alt} - h^{SLA}_{steric} + v^{h^{SLA}_{GRACE}}, \qquad (11)$$

Some notes for the GRACE data and resulting SLA should be pointed out. GRACE spherical harmonic expansions do not include degree1 harmonics, therefore in order to be consistent with the altimetric data (T/P, JASON1/2, etc.) the ones used for the latter should be employed. Furthermore, during GRACE data processing, atmospheric and barotropic models are used to eliminate atmospheric and ocean loading effects. But, since we are interested in the full ocean signal (see also the discussion in section 2.1 for the altimetry data), these need to be restored in consistency with the ones applied for the altimetric data processing (Garcia et al. 2007). Finally, GRACE data need to be corrected for the GIA where a linear term of ~1.7 mm/y is sufficient (Lombard et al. 2007).

3 Data combination

Having outlined the observation equations for the input data to be used, Eqs. (3), (5), (6), (7) and (11) we can compact them in the following form based on LSC with parameters

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{s} + \mathbf{v}, \qquad (12)$$

where the bold faced letters denote vectors, ***y*** is the vector of observations, ***x*** is the vector of parameters, ***A*** is the design matrix, ***s*** is the vector of signals and ***v*** is the vector errors. Based on the analysis of the observations to be used we can write the full observation vector

$$\mathbf{y} = \begin{bmatrix} \underset{m\times1}{\mathbf{h}^{SSH^T}} & \underset{q\times1}{\mathbf{h}^{T}_{TG}} & \underset{k\times1}{\mathbf{h}^{SLA^T}_{GRACE}} & \underset{p\times1}{\mathbf{T}^{T}} & \underset{p\times1}{\mathbf{T}^{T}_{rr}} \end{bmatrix}^T, \qquad (13)$$

where we assume to have available *m* altimetric, *q* TG, *k* GRACE and *p* GOCE observations and the vector of deterministic parameters as

$$\mathbf{x} = \begin{bmatrix} \begin{bmatrix} \cdots \\ a^{alt} \\ b^{alt} \\ \cdots \end{bmatrix}_{m\times1} & a^{TG}_{q\times1} & 0_{(k+2p)\times1} \end{bmatrix}^T. \qquad (14)$$

The signals must all be consistently related through one and the same covariance function, so we can write the matrix of signals (Barzaghi et al. 2009)

$$\mathbf{s} = \begin{bmatrix} \frac{1}{\gamma}\mathbf{T} + \boldsymbol{\zeta}^c + \mathbf{h}^{SLA}_{alt} \\ \frac{1}{\gamma}\mathbf{T} + \boldsymbol{\zeta}^c + \mathbf{H}^{ISL}_{TG} \\ \frac{1}{\gamma}\delta\mathbf{T} - \boldsymbol{\delta\zeta} \\ \mathbf{T} \\ \mathbf{T}_{rr} \end{bmatrix}. \qquad (15)$$

Note that in Eq. (15) instead of $\boldsymbol{h}^{SLA}_{steric}$ we have used the signal of the time-varying SST $\boldsymbol{\delta\zeta}$ which is equivalent. The same holds for $h^{alt^{SLA}}_{steric}$ as it is already mentioned in §2.1.

If we assume that: a) the noises of the input data are independent from one another and from ***T***, b) $\boldsymbol{\delta\zeta}$ has some isotropic covariance (CV) function over the area under study, c) $\boldsymbol{\delta\zeta}$ is independent of ***T*** and from all other noises and errors, d) the errors ***v*** are noises with either given variances or prediction errors with known CV structure, e) ***T*** has some isotropic covariance function, f) $c_n(T)$ are the degree variances of the geopotential model up to its maximum degree n_{max} and we further denote $\sigma_n(T)$ the degree variances from n_{max} to ∞, then we can write the covariance function of *T* as

$$C_{TT}\left(P,Q\right) = \left(\frac{GM}{R}\right)^2\sum_{n=0}^{n_{max}} c_n\left(T\right)\left(\frac{R^2}{r_Q r_P}\right)^{n+1} P_n\left(\cos\psi_{PQ}\right) + \left(\frac{GM}{R}\right)^2\sum_{n=n_{max}+1}^{\infty}\sigma_n\left(T\right)\left(\frac{R_B^2}{r_Q r_P}\right)^{n+1} P_n\left(\cos\psi_{PQ}\right), \qquad (16)$$

where R_B denotes depth to Bjerhammar sphere and the degree variances $\sigma_n(T)$ can be determined by the Tscherning and Rapp (1974) model. Having our fundamental CV function we can determine all auto- and cross-covariance function of all other functionals through covariance propagation (see Barzhaghi et al. 2009 for analytical expressions).

In complete analogy we can define the covariance function of the quasi-stationary SST as

$$\mathbf{C}_{\zeta^c\zeta^c} = \sum_{n=0}^{\infty} \sigma_n\left(\zeta^c\right)\left(\frac{R_B}{R}\right)^{2(n+1)} P_n\left(\cos\psi\right), \tag{17}$$

where $\sigma_n\left(\zeta^c\right)$ are the degree variances of ζ^c. Note that in all cases the analytical covariance function models should agree to the empirical values available for the area under study in order to represent the local statistical characteristics of the signal under consideration, i.e., the quasi-stationary SST in this case (Knudsen 1991). For the description of the behaviour of the degree variances given in Eq. (16), Knudsen (1987, 1992, 1993) and Knudsen and Tscherning (2006) use a 3[rd] degree Butterworth filter so that the degree variances of the MDT are given as:

$$\left(\sigma_n\left(\zeta^c\right)\right)^2 = b\left(\frac{k_2^3}{k_2^3+n^3} - \frac{k_1^3}{k_1^3+n^3}\right). \tag{18}$$

The factors b, k_1, k_2 and R_B are determined so that the analytic model fits the empirical values describing the statistical characteristics of the quasi-stationary SST in the area under study and more precisely the variance and the correlation length. Various such models of the quasi-stationary SST are available and can be employed as for instance the one by Rio (Rio and Hernandez 2004).

For the GRACE data covariance function, Eq (16) is to be used with the only difference that the signal degree variances are now expressed as

$$\sigma_n\left(\delta T\right) = R^2 \sum_{n=1}^{n_{max}} \sum_{m=0}^{n} \left\{\left(\delta\bar{C}_{nm}\right)^2 + \left(\delta\bar{S}_{nm}\right)^2\right\}, \tag{19}$$

The final covariance function that we need to determine in order to have the complete problem set-up and proceed to the estimation of the signals is that of the time-varying SST. In Knudsen (1991) a model of the signal degree variances of the $\delta\zeta$ has been introduced, so that since $\delta\zeta$ varies with time, time-dependency should enter in the computation of $\boldsymbol{C}_{\delta\zeta\delta\zeta}$ in a way that temporal correlations are given in the same way as spatial ones. Therefore, for some time-separated points $\Delta t=|t-t'|$ the covariance function can be expressed as:

$$\mathbf{C}_{\delta\zeta\delta\zeta}\left(\psi,\Delta t\right) = \begin{Bmatrix} \sum_{n=0}^{\infty}\left(\sigma_n\left(\delta\zeta\right)\right)^2 P_n\left[\cos\left(\psi+\kappa^{\delta\zeta}\Delta t\right)\right] & \text{for } \left(\psi+\kappa^{\delta\zeta}\Delta t\right)\le\pi \\ 0 & \text{for } \left(\psi+\kappa^{\delta\zeta}\Delta t\right)>\pi \end{Bmatrix}, \tag{20}$$

where $\kappa^{\delta\zeta}$ is a conversion factor representing in the case of the time-varying SST the correlation time of the signal. This should be studied and determined in each region under study, since the characteristics of $\delta\zeta$ vary significantly for each area and in open or closed sea regions (Knudsen and Tscherning 2006). The degree variances $\sigma_n\left(\delta\zeta\right)$ are determined as in Eq. (17) in order to fit the local characteristics of the time-varying SST. Since the time-varying SST is triggered by salinity and temperature variations, in-situ oceanographic data and climatology models can be used for the fit of the analytical model to empirical values.

Given the analytical expressions for the covariance functions of all observations, it is possible to proceed to the simultaneous estimation of the deterministic parameters and of the signals $\boldsymbol{\zeta}^c$, $\boldsymbol{\delta\zeta}$, $\boldsymbol{T}$, $\boldsymbol{h}^{SLA}_{non\text{-}steric}$, etc., along with their prediction errors as:

$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{C}_{yy}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{C}_{yy}^{-1}\mathbf{y}, \tag{21}$$

$$\hat{\zeta}^c\left(P\right) = \mathbf{C}_{\zeta s}\left(P,\cdot\right)\mathbf{C}_{yy}^{-1}\left(\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}\right), \tag{22}$$

$$\sigma^2_{\hat{\zeta}^c}\left(P\right) = \mathbf{C}_{\zeta^c\zeta^c}\left(P,P\right) - \mathbf{C}_{\zeta^c s}\left\{\mathbf{C}_{yy}^{-1} - \mathbf{C}_{yy}^{-1}\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^T\mathbf{C}_{yy}^{-1}\right\}\mathbf{C}_{s\zeta^c}. \tag{23}$$

$$\hat{h}^{SLA}_{STERIC}\left(P\right) = \mathbf{C}_{h^{SLA}_{STERIC}s}\left(P,\cdot\right)\mathbf{C}_{yy}^{-1}\left(\mathbf{y}-\mathbf{A}\hat{\mathbf{x}}\right), \tag{24}$$

$$\begin{aligned}\sigma^2_{\hat{h}^{SLA}_{STERIC}}\left(P\right) = {} & \mathbf{C}_{h^{SLA}_{STERIC}h^{SLA}_{STERIC}}\left(P,P\right) - \\ & \mathbf{C}_{h^{SLA}_{STERIC}s}\left\{\mathbf{C}_{yy}^{-1} - \mathbf{C}_{yy}^{-1}\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^T\mathbf{C}_{yy}^{-1}\right\}\mathbf{C}_{sh^{SLA}_{STERIC}}\end{aligned}. \tag{25}$$

The estimation of all other signals and prediction errors can be done according to Eqs. (22)-(25).

A short note on the estimation of variance components within this optimal combination scheme will be given. Given the general observation equation for LSC estimation with parameters and knowing that $v\sim(0, V_\theta=C_{vv})$ we can write C_{vv} in a form as to depend on a unknown set of so-called variance components θi. Therefore, $V_\theta = \sum_i \theta_i V_i$ and the matrices Vi come from the original data error CV matrices. Within the present combination scheme $\boldsymbol{V}_\theta$ would take the form

$$\begin{aligned}\mathbf{V}_\theta = {} & \underset{1\times m}{\boldsymbol{\theta}_{h^{alt}}}\underset{m\times m}{\mathbf{Q}_{h^{alt}}} + \underset{q\times 3}{\begin{bmatrix}\boldsymbol{\theta}_{h^{TG}} & \boldsymbol{\theta}_{H^{ISL}_{TG}} & \boldsymbol{\theta}_{h^{SLA}}\end{bmatrix}}\underset{3q\times 3q}{\begin{bmatrix}\mathbf{Q}_{h^{TG}} & & 0 \\ & \mathbf{Q}_{H^{ISL}_{TG}} & \\ 0 & & \mathbf{Q}_{h^{SLA}}\end{bmatrix}} + \\ & + \underset{1\times k}{\boldsymbol{\theta}_{h^{SLA}_{GRACE}}}\underset{k\times k}{\mathbf{Q}_{h^{SLA}_{GRACE}}} + \underset{1\times p}{\boldsymbol{\theta}_T}\underset{p\times p}{\mathbf{Q}_T} + \underset{1\times p}{\boldsymbol{\theta}_{T_{rr}}}\underset{p\times p}{\mathbf{Q}_{T_{rr}}}\end{aligned} \tag{26}$$

The variance components can then be estimated by an iterative procedure such as the *iterative almost unbiased estimation-IAUE* (Rao and Kleffe 1988). According to IAUE, we first have to compute matrix $\boldsymbol{W}$ as

$$\mathbf{W} = \mathbf{C}_{vv}^{-1} - \mathbf{C}_{vv}^{-1}\mathbf{A}\left(\mathbf{A}^T\mathbf{C}_{vv}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{C}_{vv}^{-1}, \tag{27}$$

and then estimate the variance components $\hat{\theta}$ according to $\hat{\boldsymbol{\theta}} = \mathbf{J}^{-1}\mathbf{k}$ or $\hat{\boldsymbol{\theta}} = \mathbf{J}^{+}\mathbf{k}$ depending on whether matrix $\boldsymbol{J}$ can be inverted. Matrices $\boldsymbol{J}$ and $\boldsymbol{k}$ can be analytical determined from the data error CV matrices

$$J_{ij} = \mathrm{tr}\left[\mathbf{W}\mathbf{C}_{v_i v_i}\mathbf{W}\mathbf{C}_{v_j v_j}\right], \tag{28}$$

$$k_i = \hat{\mathbf{v}}^T \mathbf{C}_{vv}^{-1} C_{v_i v_i} \mathbf{C}_{vv}^{-1} \hat{\mathbf{v}}. \tag{29}$$

After the first variance components have been estimated, the iterations can be carried out according to

$$\hat{\theta}_i^{\alpha} = \frac{\hat{\theta}_i^{\alpha-1} \hat{\mathbf{v}}^T \mathbf{C}_{vv}^{-1} C_{v_i v_i} \mathbf{C}_{vv}^{-1} \hat{\mathbf{v}}}{\mathrm{tr}\left\{\mathbf{W} C_{v_i v_i}\right\}}. \tag{30}$$

where $\hat{\theta}_i^{\alpha-1}$ is the previous estimate and $\hat{\theta}_i^{\alpha}$ is the new one. When their difference is smaller than a certain threshold ε, so that $\left|\hat{\boldsymbol{\theta}}^{\alpha} - \hat{\boldsymbol{\theta}}^{\alpha-1}\right| < \varepsilon$, then convergence is achieved, the variance components are estimated and the iterations can stop.

4 Conclusions

A detailed combination scheme of satellite altimetry, tide-gauge, GRACE and GOCE data for the determination of various signals both static and variable has been presented. The latter, can be the disturbing potential, its second order derivatives, geoid heights, steric and non-steric sea level variations and the stationary and time-variable sea surface topography. It is clear that the versatility of LSC with parameters allows including other observations too, like gravity anomalies, salinity and temperature from oceanographic measurements, hydrological models, etc. Therefore, the proposed combination scheme can be extended easily as new data sources become available, so that better estimates will be derived. Another advantage of the combination strategy outlined is that rigorous signal estimation errors can be derived along with the possibility of variance component estimation so that the covariance matrices and errors models can be calibrated.

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