Combination Schemes for Local Orthometric Height Determination from GPS Measurements and Gravity Data

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Abstract. One of the most interesting and challenging tasks in the field of geodetic surveying is the accurate determination of orthometric heights from GPS measurements taking into account leveling data and additional gravity field information. This paper focuses on the presentation of the currently available various solution strategies which are then properly applied. The first method is based on the integrated geodetic model, where gravity field parameters are treated as signals. A second solution is based on a combination scheme employing least squares collocation as the optimal heterogeneous combination method for gravity and height data. Another method is the spectral domain equivalent of least squares collocation, namely the Multiple Input Multiple Output System Theory, where gravity and height data are treated as stochastic signals with full variance covariance information. The last method consists in a polynomial interpolation model of various orders expressing different geoid representations.

Keywords. Height combination, collocation, integrated methods, MIMOST, polynomial interpolation.

1 Introduction

The current availability of ever more accurate regional and local geoid models, the dramatic improvement in Global Geopotential Model (GGM) determination and the expected impact of the GOCE mission towards a cumulative geoid error of ±1 cm to degree and order 200, make height combination schemes of major importance for a variety of applications in geodesy. The combination of various types of heights has been a topic of geodetic research for nearly forty years, originating from the pioneering work of Krarup (1969), who first presented the integrated approach of geodetic data adjustment. From that idea stem the origins of the leading estimation principle in modern geodetic research, i.e., that of least squares collocation (LSC), which due to the use of many and various types of data and through the use of Fourier transforms (FT) introduced the use of system theory with geodetic data. These ideas are presented in this work in the frame of geoid height combination, when both gravimetric and GPS/Leveling geoid heights are available.

2 The integrated approach

Integrated geodesy has been introduced for the rigorous adjustment of observations with both geometric and gravimetric information using precise mathematical models. Furthermore, integrated geodesy is a method for the adjustment of observations depending not only on discrete parameters but also on unknown functions. Specific applications related to the estimation of orthometric heights from GPS baselines, leveling and gravity observations have been presented by Hein (1985), Hein et al. (1988) and Hatjidakis and Rossikopoulos (2002).

The observational data, considered in the integrated approach, can be GPS baselines and coordinates, orthometric heights, geoidal undulations, gravity anomalies, potential differences as well as data of any functional related to the Earth's gravity field. As more simply endorsed in the literature, we have the equations of geodetic heights h_i^{GPS}

$$h_i^{GPS} = H_i + N_i + v_i^h \qquad \text{(Eq.1)}$$

which result from the analysis of GPS observations. In the same way, the equations of orthometric heights H_i^{LEV}

$$H_i^{LEV} = H_i + v_i^H \qquad \text{(Eq.2)}$$

and geoid heights N_i^{GEO}

$$N_i^{GEO} = N_i + \delta N_i + v_i^{\zeta} \qquad \text{(Eq.3)}$$

are determined. In Eq. 3, the parameter δN_i describes all possible datum inconsistencies and other systematic effects in the data sets.

All observations, which can be different at every

Table 1. The observations equations for orthometric height determination in the integrated adjustment approach.

GPS coordinates $\mathbf{r}_i\,(X_i, Y_i, Z_i)$:	$\mathbf{r}_i - \mathbf{r}_i^o + h_i^o \mathbf{m}_i = \mathbf{m}_i H_i + \mathbf{m}_i N_i$
or	$\mathbf{r}_i - \mathbf{r}_i^o = \mathbf{m}_i \delta H_i + \mathbf{m}_i N_i$
where $\mathbf{m}_i = \begin{bmatrix} \cos\varphi_i \cos\lambda_i \\ \cos\varphi_i \sin\lambda_i \\ \sin\varphi_i \end{bmatrix}$ is the unit vector to the reference ellipsoid.	
GPS baselines:	$\mathbf{r}_{ij} - \mathbf{r}_{ij}^o = -\mathbf{m}_i \delta H_i + \mathbf{m}_j \delta H_j - \mathbf{m}_i N_i + \mathbf{m}_j N_j$
or	$\mathbf{r}_{ij} - \mathbf{r}_{ij}^o - h_i^o \mathbf{m}_i + h_j^o \mathbf{m}_j = -\mathbf{m}_i H_i + \mathbf{m}_j H_j - \mathbf{m}_i N_i + \mathbf{m}_j N_j$
Geodetic heights: $h_i^{GPS} = H_i + N_i + v_i^h = H_i + \frac{1}{\gamma_i} T_i + v_i^h$	
Orthometric height differences: $\Delta H_{ij} = H_j - H_i$	
Gravity values:	$g_i = \gamma_i^o + \mathbf{m}_i^T \mathbf{M}(\mathbf{r}_i - \mathbf{r}_i^o) + \mathbf{m}_i^T\, gradT_i$
or	$g_i = \tilde{\gamma}_i^o + \mathbf{a}_g^T \mathbf{m}_i \delta H_i + \delta g_i - \frac{1}{\gamma_i^o} \mathbf{a}_g^T \mathbf{m}_i T_i$
where $\tilde{\gamma}_i^o = \gamma_i^o - N_i^o \mathbf{a}_g^T \mathbf{m}_i$ and the vector $\mathbf{a}_g$ depends on the Marussi matrix **M**.	
Geoid heights: $N_i^{GEO} = N_i + \delta N_i + v_i = \frac{1}{\gamma_i} T_i + \delta N_i + v_i$	
Potential differences: $\Delta W_{ij} = -\gamma_i^o H_i + \gamma_j^o H_j + T_j - T_i + v_{ij}$	

point, can be analyzed simultaneously with the general least squares collocation model

$$\mathbf{w} = \mathbf{A}\,\mathbf{x} + \mathbf{G}\,\mathbf{s} + \mathbf{v} \quad \text{(Eq.4)}$$

where $\boldsymbol{x}$ contains the orthometric heights ($x_i = H_i$, the deterministic parameters), $\boldsymbol{s}$ contains the geoid heights ($s_i = N_i = T_i / \gamma_i$, the stochastic parameters) and $\boldsymbol{v}$ are the observational errors. The adjustment problem is twofold, i.e., estimation with respect to $\boldsymbol{x}$ and prediction with respect to $\boldsymbol{s}$ and $\boldsymbol{v}$. For the stochastic parameters it is assumed that their means

$$E\{\mathbf{s}\} = \boldsymbol{\mu},\; E\{\mathbf{v}\} = \mathbf{0} \quad \text{(Eq.5)}$$

and the covariance matrices are given as

$$E\{\mathbf{v}\mathbf{v}^T\} = \mathbf{C} = \begin{bmatrix} \sigma_h^2 \mathbf{Q}_h & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_H^2 \mathbf{Q}_H & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_N^2 \mathbf{Q}_N \end{bmatrix}$$

$$E\{(\mathbf{s}-\boldsymbol{\mu})(\mathbf{s}-\boldsymbol{\mu})^T\} = \mathbf{K} \quad \text{(Eq.6)}$$

and $E\{(\mathbf{s}-\boldsymbol{\mu})\mathbf{v}^T\} = \mathbf{0}$. The covariance matrix of signals $\boldsymbol{K}$ is obtained from the covariance function *K(P,Q)* of the disturbing potential at two different points *P* and *Q*, by applying the law of covariance propagation to the functionals relating the signals with the disturbing potential. For example, the geoid height covariance function is given as

$$\mathbf{K}_N(P,Q) = \frac{1}{\gamma_P \gamma_Q} \mathbf{K}(P,Q) \quad \text{(Eq.7)}$$

Initially, an empirical covariance function is determined from the gravity anomalies. Local covariance models for the disturbing potential on the local plane extended to the subspace above it and the corresponding functions of the gravity anomaly are given in Table 2. The adjustment of observations is carried out by applying the least squares principle (Dermanis, 1987; Dermanis and Fotiou 1992)

$$\mathbf{v}^T \mathbf{C}^{-1} \mathbf{v} + \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s} = \min \quad \text{(Eq.8)}$$

which leads to best linear unbiased estimates for the deterministic parameters $\boldsymbol{x}$ and best linear unbiased predictions for the stochastic ones $\boldsymbol{s}$, $\boldsymbol{v}$.

3 The model function approach

Let us assume that geoid height data is not available in the region of a GPS network, but some orthometric heights are known. The observation equations

Table 2: Local covariance models

	For gravity anomaly $K_{\Delta g}(S)$	For disturbing potential K(S,z)
Exponential model	$\sigma_g^2 e^{-\frac{S^2}{2d^2}}$	$C_{\tilde{E}}(1,0)$
Reilly model	$\sigma_g^2 (1-\frac{S^2}{2d^2}) e^{-\frac{S^2}{2d^2}}$	$C_R(1,0)$
Moritz model	$\sigma_g^2 d^3 \frac{2d^2 - S^2}{2\sqrt{(S^2+d^2)^5}}$	$\sigma_g^2 \frac{d^3}{2\sqrt{S^2+(z+d)^2}}$
Poisson model	$\sigma_g^2 d^5 \frac{6d^2 - 9S^2}{2\sqrt{(S^2+d^2)^7}}$	$\sigma_g^2 \frac{(z+d)d^4}{6\sqrt{(S^2+(z+d)^2)^3}}$

where

$$C_E(q,m) = \sigma_g^2 (\frac{d}{\sqrt{2}})^{1-q} \frac{\rho^m}{m!} \sum_{k=0}^{l} \left\{ \Gamma(\frac{m+q+k+1}{2}) {}_1F_1(\frac{m+q+k+1}{2}; m+1; -\rho^2) \frac{(-\zeta)^k}{k!} \right\}$$

$$C_R(q,m) = \frac{d^2}{2} C_E(q,m),\ \rho = \frac{S}{\sqrt{2}d},\ \ \zeta = \frac{(z_i + z_j)\sqrt{2}}{d}$$

$${}_1F_1(a;c;x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \ldots$$

$$\Gamma(x) = (x-1)! \text{ when } x \succ 1\ \forall x \in Z \quad \text{or} \quad \Gamma(x+\frac{1}{2}) = \frac{1\ 3\ 5\ \ldots\ (2\ x-1)}{2^x}\Gamma(\frac{1}{2})$$

S is the horizontal distance between two signals, d the correlation length and z is the height from reference level of covariance function.

for each point with known orthometric height are written

$$h_i^{GPS} = H_i + N_i + v_i^h$$

$$H_i^{LEV} = H_i + v_i^H \ , \quad i = 1,2,\ldots,n \qquad \text{(Eq.9)}$$

or in equivalence

$$u_i = h_i^{GPS} - H_i^{LEV} = N_i + v_i \qquad \text{(Eq.10)}$$

where $v_i = v_i^h - v_i^H$ is the total error. The most usual method in surveying applications is based on the calculation of geoid heights using an analytic function in the form

$$N_i = \sum_{k=0}^{m} \sum_{l=0}^{m} \alpha_{kl} x_i^k y_i^l \qquad \text{(Eq.11)}$$

The observation equation becomes

$$u_i = h_i^{GPS} - H_i^{LEV} = \sum_{k=0}^{m} \sum_{l=0}^{m} \alpha_{kl} x_i^k y_i^l + v_i \qquad \text{(Eq.12)}$$

and for all the points in matrix notation it becomes $\mathbf{u} = \mathbf{Fa} + \mathbf{e}$, where we can use the principle $\mathbf{e}^T\mathbf{e} = \min$ for the exact interpolation or, for the smoothing interpolation, $\mathbf{e}^T\mathbf{M}^{-1}\mathbf{e} = \min$, where $\mathbf{M} = \sigma_h^2\mathbf{Q}_h + \sigma_H^2\mathbf{Q}_H$. Therefore, in this case geoid heights are treated through a parametric surface.

4 A hybrid interpolation approach

Let us assume, in contrast to the model presented in §3, that geoid heights are available for all GPS network benchmarks, and some orthometric heights are also known. For these points we have the three observation equations

$$h_i^{GPS} = H_i + N_i + v_i^h$$

$$H_i^{LEV} = H_i + v_i^H$$

$$N_i^{GEO} = N_i + \delta N_i + v_i^{\zeta} \qquad \text{(Eq.13)}$$

or in equivalence

$$h_i^{GPS} - H_i^{LEV} - N_i^{GEO} = \delta N_i + v_i \quad \text{(Eq.14)}$$

where $v_i = v_i^h - v_i^H - v_i^N$. The correction term δN_i can be decomposed in the form (Kotsakis and Sideris, 1999)

$$\delta N_i = \mathbf{f}_i^T \mathbf{a} + s_i \quad \text{(Eq.15)}$$

where $\mathbf{f}_i^T \mathbf{a}$ and s_i are the trend and signal components, and by using matrix notation in order to combine the points of the network with the triple information, we obtain

$$\mathbf{u} = \mathbf{Fa} + \mathbf{s} + \mathbf{v} \quad \text{(Eq.16)}$$

The adjustment is carried out by applying the least squares principle

$$\mathbf{v}^T \mathbf{M}^{-1} \mathbf{v} + \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s} = \min \quad \text{(Eq.17)}$$

where $\mathbf{M} = \sigma_h^2 \mathbf{Q}_h + \sigma_H^2 \mathbf{Q}_H + \sigma_N^2 \mathbf{Q}_N$. Then, the integrated adjustment model

$$h_i^{GPS} = H_i + N_i + v_i^h$$

$$H_i^{LEV} = H_i + v_i^H$$

$$\tilde{N}_i^{GEO} = N_i^{GEO} - \delta \hat{N}_i = N_i + v_i^{\zeta} \quad \text{(Eq.18)}$$

can be used to the overall "observations" of the network for the estimation of orthometric and geoid heights, where the corrections $\delta \hat{N}_i = \mathbf{f}_i^T \hat{\mathbf{a}} + \hat{s}_i$ are calculated by using the values $\hat{\mathbf{a}}$ and $\hat{s}_i$.

5 System theory in gravity field modeling

In this section, combination schemes of heterogeneous data in the frequency domain are presented, while a specific example of gravimetric and GPS/Leveling geoid heights is given. Moreover, the similarities and differences between system theory and LSC are outlined, in order to show the physical relation between all methods presented in this work.

Spectral methods, and FT in particular, have been extensively used since the beginning of the 70's for the solution of the classical boundary value problems of physical geodesy. The key concept for the utilization of FT in geodetic problems lies in the representation of well-known integral formulas (e.g., Stokes' and Vening-Meinesz integrals for the prediction of geoid heights from gravity anomalies and deflections of the vertical, respectively) as convolution integrals. Since in the spectral domain the convolution of some input signals is replaced by simple multiplication of their spectra, FT and Fast Fourier Transforms (FFT) have been used mainly due to the high-efficiency in terms of time that they offer compared to the usual integral methods of solving geodetic boundary value problems. Despite the gain in processing time, FFT methods carry some disadvantages, among which the main ones are: a) the need for regularly spaced (i.e., gridded) data, b) the inability to predict the estimation error for the output signal and c) the prerequisite of having a single input and a single output signal. On the other hand, the leading estimation method in physical geodesy, i.e., LSC, which was previously discussed in the frame of height combination schemes, allows the use of multiple input signals and irregularly distributed data, while it provides an optimal, under the Wiener-Kolmogorov principle, estimate of the output signal with simultaneous estimation of the full variance-covariance matrix of the output signal error (Moritz 1980).

Nevertheless, especially in modern day geodetic applications with the hundreds of thousands of altimetric, gravimetric and space borne gravity field related data, the application of LSC has become cumbersome. Therefore, a frequency domain equivalent to LSC has been developed employing system theory. The latter has been traditionally used in signal processing and signal transmission methods as well as in various applications of electrical engineering. The first, who proposed a solution of geodetic boundary value problems in the frequency domain employing system theory was Sideris (1996), who presented the general scheme for the use of a system with multiple input and multiple outputs (Multiple Input Multiple Output System Theory – MIMOST). Numerical solutions and examples of using MIMOST methods for the estimation of geoid heights, gravity anomalies, deflections of the vertical, the quasi-stationary sea surface topography from heterogeneous noisy data as well as in combined gravimetric and GPS geoid solutions have been presented in several papers (see, e.g., Andritsanos 2000; Andritsanos et al. 2001, 2004; Andritsanos and Tziavos 2002; Vergos et al. 2005).

A MIMOST system with two input signals and a single output is presented in Figure 1, where an example of gravimetric and GPS/Leveling geoid heights combination is presented for the prediction of combined geoid heights. In many cases, since specific information for the input signal noise is not available, simulated noises are generated as input error under the assumption of white noise. It should

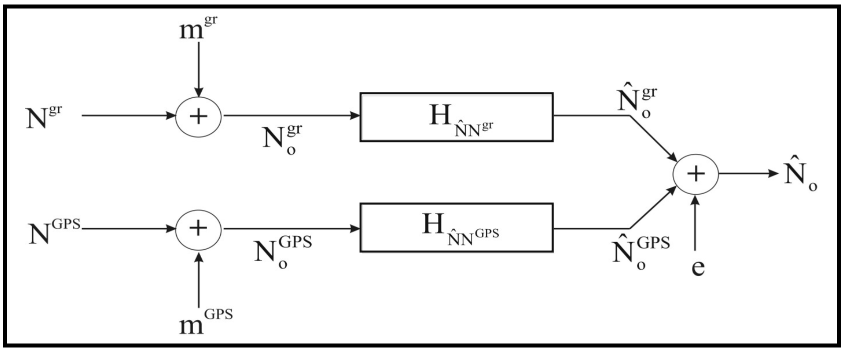

Figure 1: A dual-input single output system for the prediction of geoid heights.

be noted that as shown by Andritsanos et al. (2001) in the case of repeat altimetric missions an estimation of the input error Power Spectral Density (PSD) function can be directly evaluated using this successive information. The final solutions and the error PSD function of the MIMOST method are calculated according to the following equations:

$$\begin{aligned}\hat{\mathbf{N}}_o = & \begin{bmatrix} \mathbf{H}_{\hat{N}N^{gr}} & \mathbf{H}_{\hat{N}N^{GPS}} \end{bmatrix} \\ & \left(\begin{bmatrix} \mathbf{P}_{N_o^{gr}N_o^{gr}} & \mathbf{P}_{N_o^{gr}N_o^{GPS}} \\ \mathbf{P}_{N_{oa}N_o^{gr}} & \mathbf{P}_{N_o^{GPS}N_o^{GPS}} \end{bmatrix} - \begin{bmatrix} \mathbf{P}_{m^{gr}m^{gr}} & 0 \\ 0 & \mathbf{P}_{m^{GPS}m^{GPS}} \end{bmatrix} \right) \\ & \begin{bmatrix} \mathbf{P}_{N_o^{gr}N_o^{gr}} & \mathbf{P}_{N_o^{gr}N_o^{GPS}} \\ \mathbf{P}_{N_{oa}N_o^{gr}} & \mathbf{P}_{N_o^{GPS}N_o^{GPS}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{N}_o^{gr} \\ \mathbf{N}_o^{GPS} \end{bmatrix}\end{aligned} \quad \text{(Eq.19)}$$

$$\begin{aligned}\mathbf{P}_{\hat{e}\hat{e}} = & \left\{ \begin{bmatrix} \mathbf{H}_{\hat{N}N^{gr}} & \mathbf{H}_{\hat{N}N^{GPS}} \end{bmatrix} \right. \\ & \left(\begin{bmatrix} \mathbf{P}_{N_o^{gr}N_o^{gr}} & \mathbf{P}_{N_o^{gr}N_o^{GPS}} \\ \mathbf{P}_{N_{oa}N_o^{gr}} & \mathbf{P}_{N_o^{GPS}N_o^{GPS}} \end{bmatrix} - \begin{bmatrix} \mathbf{P}_{m^{gr}m^{gr}} & 0 \\ 0 & \mathbf{P}_{m^{GPS}m^{GPS}} \end{bmatrix} \right) - \\ & - \begin{bmatrix} \hat{\mathbf{H}}_{\hat{N}_oN_o^{gr}} & \hat{\mathbf{H}}_{\hat{N}_oN_o^{GPS}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{N_o^{gr}N_o^{gr}} & \mathbf{P}_{N_o^{gr}N_o^{GPS}} \\ \mathbf{P}_{N_{oa}N_o^{gr}} & \mathbf{P}_{N_o^{GPS}N_o^{GPS}} \end{bmatrix} \\ & \left(\begin{bmatrix} \mathbf{H}^*_{\hat{N}N^{gr}} \\ \mathbf{H}^*_{\hat{N}N^{GPS}} \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{H}}^*_{\hat{N}_oN_o^{gr}} \\ \hat{\mathbf{H}}^*_{\hat{N}_oN_o^{GPS}} \end{bmatrix} \right) \begin{bmatrix} \hat{H}_{\hat{N}_oN_o^{gr}} & \hat{H}_{\hat{N}_oN_o^{GPS}} \end{bmatrix} \\ & \begin{bmatrix} \mathbf{P}_{m^{gr}m^{gr}} & 0 \\ 0 & \mathbf{P}_{m^{GPS}m^{GPS}} \end{bmatrix} \begin{bmatrix} \mathbf{H}^*_{\hat{N}N^{gr}} \\ \mathbf{H}^*_{\hat{N}N^{GPS}} \end{bmatrix}\end{aligned} \quad \text{(Eq.20)}$$

where $\hat{\mathbf{N}}_o$ is the combined geoid estimation, N^{gr} and N^{GPS} are the pure gravimetric and GPS/Leveling signals respectively, N_o^{gr} and N_o^{GPS} are the gravimetric and GPS/Leveling observations, m^{gr} and m^{GPS} are the input noises, H_{xy} is the theoretical operator that connects the pure input and output signals, $\hat{\mathbf{H}}_{x_oy_o}$ is the optimum frequency impulse response function, $\mathbf{P}_{\hat{e}\hat{e}}$ is the error PSD function, e is the noise of the output signal and the * denotes complex conjugate of the matrix under consideration.

If we substitute the vector of observation and estimation signals with

$$\mathbf{Y}_o = \begin{bmatrix} N_o^{gr} \\ N_o^{GPS} \end{bmatrix}; \quad \mathbf{X}_o = \begin{bmatrix} N_o \end{bmatrix}, \quad \text{(Eq.21)}$$

then Eqs. 19 and 20 can be written in matrix notation as

$$\begin{aligned}\hat{\mathbf{X}}_o &= \hat{\mathbf{H}}_{X_oY_o}\mathbf{Y}_o = \mathbf{P}_{XY}\mathbf{P}^{-1}_{Y_oY_o}\mathbf{Y}_o = \\ &= \mathbf{H}_{XY}\left(\mathbf{P}_{Y_oY_o} - \mathbf{P}_{mm}\right)\mathbf{P}^{-1}_{Y_oY_o}\mathbf{Y}_o\end{aligned} \quad \text{(Eq.22)}$$

$$\begin{aligned}\mathbf{P}_{\hat{e}\hat{e}} = & \left[\mathbf{H}_{XY}\left(\mathbf{P}_{Y_oY_o} - \mathbf{P}_{mm}\right) - \right. \\ & \left. - \hat{\mathbf{H}}_{X_oY_o}\mathbf{P}_{Y_oY_o} \right]\left(\mathbf{H}^{*T}_{XY} - \hat{\mathbf{H}}^{*T}_{X_oY_o}\right) + \\ & + \hat{\mathbf{H}}_{X_oY_o}\mathbf{P}_{mm}\mathbf{H}^{*T}_{XY}\end{aligned} \quad \text{(Eq.23)}$$

where and the theoretical operator impulse response function is

$$\mathbf{H}_{XY} = \mathbf{P}_{XY}\mathbf{P}^{-1}_{YY} \quad \text{(Eq.24)}$$

In order to see the equivalence with space domain least squares collocation, lets assume that we have a stationary, isotropic random input signal described by the vector

$$\mathbf{y} = \begin{bmatrix} N^{gr} \\ N^{GPS} \end{bmatrix} \quad \text{(Eq.25)}$$

and that there exists a linear estimator ***h(x,y)*** (represented by ***h*** for simplicity) which relates the input signal ***y*** with the output signal ***x***, i.e.,

$\mathbf{x}=\mathbf{h}\mathbf{y}$ (Eq.26)

If we denote the error vector by ***e*** then its covariance matrix will be given as:

$$\mathbf{C}_{\hat{e}\hat{e}} = E\left\{\mathbf{e}\mathbf{e}^{\mathrm{T}}\right\} = \mathbf{h}E\left\{\mathbf{Y}\mathbf{Y}^{\mathrm{T}}\right\}\mathbf{h}^{\mathrm{T}} - E\left\{\mathbf{X}\mathbf{Y}^{\mathrm{T}}\right\}\mathbf{h}^{\mathrm{T}} - \mathbf{h}E\left\{\mathbf{Y}\mathbf{X}^{\mathrm{T}}\right\} + E\left\{\mathbf{X}\mathbf{X}^{\mathrm{T}}\right\} \quad \text{(Eq.27)}$$

where *E{}* denotes expectation. From Eq. 27, taking into account that all our signals are centered ($E\{\cdot\}=0$) and that $\mathbf{C}_{(\cdot)(\cdot)}= E\{(\cdot)(\cdot)^T\}$, after some simple substitutions we arrive at the following expression for the error covariance matrix of the output signal

$$\mathbf{C}_{\hat{e}\hat{e}} = \mathbf{C}_{XX} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\mathbf{C}_{YX} + \left(\mathbf{h} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\right)\mathbf{C}_{YY}\left(\mathbf{h} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\right)^{T} \quad \text{(Eq.28)}$$

Eq. 28 shows that the error covariance matrix of the predicted signal is composed by two parts, one that depends on the linear operator ***h*** (lets denote it as $\mathbf{A}_2 = \left(\mathbf{h} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\right)\mathbf{C}_{YY}\left(\mathbf{h} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\right)^{T}$) and another one that is independent of ***H*** (we denote it as $\mathbf{A}_1 = \mathbf{C}_{XX} - \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1}\mathbf{C}_{YX}$). The latter means that matrix $\boldsymbol{A}_1$ does not change for every possible linear prediction and every possible linear operator ***h***. According to Moritz (1980) in order to achieve the best unbiased minimum variance linear estimation of signal ***X*** from ***Y***, matrix $\boldsymbol{A}_2$ should be equal to zero, which holds if our linear operator is given as

$$\mathbf{h} = \mathbf{C}_{XY}\mathbf{C}_{YY}^{-1} \quad \text{(Eq.29)}$$

Comparing Eqs. 24 and 29 we can easily verify that they are in fact the same with $\boldsymbol{h}(\mathrm{x},\mathrm{y}) \xrightarrow{\mathscr{F}} \boldsymbol{H}_{\mathrm{XY}}$, i.e., that ***h*** and $\boldsymbol{H}_{XY}$ form a FT pair. Given that, and comparing Eqs. 23 and 28 we can easily deduce the similarity between MIMOST and LSC. An extensive comparison between the two methods is given in Sansò and Sideris (1997).

Conclusions

A complete overview of the basic concepts in height adjustment has been presented, starting from the concept of the integrated approach in geodesy, least squares collocation, multiple input multiple output system theory and a hybrid approach treating both deterministic and stochastic errors, the first representing datum inconsistencies and the later random errors in the height data to be combined. Moreover, a comparison between least squares collocation and its frequency-domain equivalent of multiple-input multiple-output system theory has been presented showing that the two methods are practically identical.

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