Combination of gravity, altimetry and GPS/Leveling data for the numerical solution of altimetry-gravimetry boundary value problems

R.S. Grebenitcharsky, G.S. Vergos, and M.G. Sideris
Department of Geomatics engineering, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada, T2N 1N4

Abstract. Using GEOSAT GM altimetry data, shipborne gravity, land gravity and GPS/leveling data, a numerical solution for the fixed altimetry-gravimetry boundary value problem (AGBVP) II is evaluated and tested. Two types of solutions are applied – with and without smoothness conditions along the coastline. Data available in the area of Newfoundland, eastern coastal region of Canada, are used. A comparison with a standard estimation method for the efficient combination of heterogeneous data, namely multiple input, multiple output system theory (MIMOST), is carried out. Conclusions for the combination of different data types and smoothness conditions along the coastline are drawn.

Keywords. Geoid determination, marine geoid, satellite altimetry, sea surface heights, shipborne gravimetry, combination, MIMOST.

1 Introduction

Altimetry-gravimetry boundary value problems (AGBVPs) provide the frame in which the combination of different types of data is possible. Historically, the first type of AGBVPs appeared in Sanso (1981) and Arnold (1983), known later as AGBVP I. Another type of AGBVP formulated in Holota (1980) is used when gravity data are available at sea, known later as AGBVP II. Holota treated both problems in a linearized form in Holota (1983a and 1983b). Later, with the availability of GPS/levelling data, a new AGBVP appeared, called AGBPV III (Lehmann, 1999). All AGBVPs consist of two different types of geodetic boundary value problems (GBVPs), one on land and one at sea. They depend on the type of boundary surface and boundary conditions (measurements) used. In Sanso (1993), different types of GBVPs are described both on land and at sea. At sea we have only one type of GBVP - *Dirihlet* BVP with known boundary surface. On land two types of GBVPs are possible: the classical Molodensky GBVP with completely unknown boundary surface and the scalar free Molodensky GBVP with known geodetic latitude and longitude for every point. The first type of BVP, called vectorial free classical Molodensky BVP, has an important role from a theoretical point of view because it has more general character. From the application point of view, astronomical observations are necessary to solve this GBVP. This fact causes complications in the application of the classical Molodensky BVP. In practice, the general formulation of AGBVPs given in Lehmann (1999) is more suitable. It shows that AGBVPs consist of two parts: scalar free (unknown boundary surface) on land and fixed (known boundary surface) at sea.

Using GPS/levelling on land, the boundary surface becomes known (AGBVP III). In this case, we can use as boundary either the Earth's surface or the geoid from GPS/levelling. The Earth's surface has two main disadvantages – it is not an equipotential surface and it is rougher than the geoid. Usually, we have gravity anomalies on the geoid, which means that they can be transformed to gravity disturbances using GPS/levelling, and in this case, we will have the same type of data (for AGBVP II) both on land and at sea. Having gravity disturbances on the geoid, both for land and sea, we are able to apply Hotine's formula to derive the geoid. The main problem in this approach is the proper application of downward continuation, especially in mountainous areas.

Compatibility conditions along the coastline for a solution of AGBVPs have been discussed by Svensson. The addition of compatibility conditions along the coastline may cause the problem to become well posed but it is questionable if such conditions upon the data are realistic for practical use (Svensson, 1988). To answer this question, the effect of compatibility (smoothing) conditions on same type boundary conditions (gravity anomalies) has been investigated in Grebenitcharsky and Sideris (2001a). For a numerical solution of the fixed AGBVP II, combining different types of data with compatibility (smoothing) conditions along the coastline will increase the regularity of boundary surface and boundary conditions in coastal zones. Variational methods for the solution of GBVPs (Holota, 1997) are applied in the solution of AGBVPs very often. Finally, the conclusion given in Rektorys (1977) that "if all data of the considered problem (including the boundary) are sufficiently smooth, then the weak solution is the classical solution of the considered problem" justifies the reason to investigate the smoothing procedure along the coastline for the fixed AGBVP II, transformed to the Neumann boundary value problem.

The task of this paper is thus *to investigate a numerical solution of the fixed altimetry-gravimetry boundary value problem II using GPS/levelling and gravity data on land together with altimetry and shipborne data at sea; and to investigate the effects of applying smoothing conditions along the coastline on geoid modelling*.

The area under study is 43° ≤ φ ≤ 57° and 298° ≤ λ ≤ 312° (Newfoundland, Eastern Canada). The following data are used: free air gravity anomalies on land from the Geological Survey of Canada; GPS/Levelling data on benchmarks from the Geodetic Survey Division; shipborne free-air gravity anomalies from the Geological Survey of Canada; and GEOSAT geodetic mission (GM) satellite altimetry data at sea from NOAA (NOAA, 1997).

The EGM96 geopotential model is used as reference field. The global quasi-stationary sea surface topography (QSST) model derived from the simultaneous EGM96 adjustment complete to degree and order 20 is used for the reduction of the data from the sea surface to the geoid.

To assess the accuracy of the numerical solution, a comparison with the Canadian geoid CGG2000 is performed. Also, a comparison with a Multiple Input Multiple Output System Theory (MIMOST) (Sideris, 1996) solution is done to assess the accuracy of the numerical solution with respect to a standard heterogeneous data combination method.

2 Numerical Solution of AGBVP II

The mixed AGBVP II in spherical approximation (Sanso, 1993) is given in eq. 1 with the following notation: T is the disturbing potential, Δ is the Laplace's operator, r is the radial distance from the center of the sphere, R is the radius of the sphere, Δg are the gravity anomalies and δg are the gravity disturbances.

$$\left| \begin{array}{ll} \Delta T = 0 & \\ -\dfrac{\partial T}{\partial r} - \dfrac{2}{R} T = \Delta g & \text{on land} \\ -\dfrac{\partial T}{\partial r} = \delta g & \text{at sea} \end{array} \right. \tag{1}$$

The boundary condition on land can be reformulated using geoid heights from GPS/leveling and Bruns's equation as follows:

$$-\frac{\partial T}{\partial r} = \delta g = \Delta g + \frac{2G}{R} N \tag{2}$$

where G is mean gravity and N is the geoid height from GPS/Leveling.

After reformulation of the land boundary condition and applying smoothing conditions along the coastline (to have regular boundary surface and regular boundary conditions, i.e., data) AGBVP II is transformed to an Neumann boundary value problem (Rektorys, 1977). Evaluating by 2D FFT the spherical Hotine convolution integral with 50 kilometers integration radius, a numerical solution for the fixed AGBVP II is obtained.

3 Smoothing (Compatibility) Conditions along the Coastline

For AGBVP I and AGBVP II, it has been shown (Svensson, 1998) that with the introduction of additional compatibility conditions on coastline, both problems become normal solvable (satisfying the Fredholme alternative). To achieve this, Svensson (1983) introduced the new approach of pseudodifferential operators. Using the theory of pseudodifferential operators it is possible not only to reformulate existing AGBVPs but to apply the compatibility conditions along the coastline (Grebenitcharsky and Sideris, 2001b). Together with new pseudodifferential operators, which are mappings between different Sobolev spaces, a new form of AGBVP I and AGBVP II is possible. The derivation of the compatibility conditions on the coastline shows that they correspond to the case when the boundary conditions (measurements) for land and sea are consistent along the coastline. To answer the question "if compatibility conditions upon the data are realistic for practical use" (Svensson, 1988), the effect on geoid of data and boundary surface smoothing along the coastline should be investigated.

From a theoretical point of view, the regularity of the boundary surface and the boundary conditions is the necessary condition for the numerical solution of AGBVP II to be close enough to the solution of the classical Neumann boundary value problem

(Rektorys, 1977). From an application point of view, existing discrepancies between different data with different resolution and accuracy could be smoothed before the application of the numerical solution. The regularity of data and boundary surface is closely related to their derivatives up to infinit order and therefore, wavelet transforms could be used as multiscale differential operators (Mallat, 1998). The necessary and sufficient condition for a wavelet transform to be an n-order multiscale differential operator is the corresponding wavelet to have *n* number of vanishing moments. Wavelet decomposition and reconstruction could be used to detect and smooth irregularities along the coastline (Grebenitcharsky and Sideris, 2001a).

The properties of wavelets to give not only the frequencies of a signal but also their spatial distribution in different scales can be used to detect discrepancies between different data along the coastline. A wavelet decomposition and reconstruction can be used to impose smoothness compatibility conditions (Grebenitcharsky and Sideris, 2001b) on data and the boundary along the coastline. After the decomposition up to a certain level, we could eliminate irregularities on the coastline in the high frequency part of the decomposition. This is equivalent to imposing constrains on the n^{th} derivatives (i.e., we use smoothness conditions). The effect of smoothing along the coastline on the final geoid solution is investigated not only for Hotine's solution but for a solution by MIMOST. The effect of smoothing conditions along the coastline on the boundary surface, on boundary conditions and on transformed boundary conditions is investigated.

4 Other Solutions

To assess the results of the numerical Hotine solution, they were compared to the following three solutions:

Solution 1: Evaluation by 2D FFT of the spherical Stokes integral kernel after simple merging of gravity anomalies on land and at sea.

Solution 2: Application of multiple input, multiple output system theory (MIMOST) method (Andritsanos et al. 2000) for combination of gravity anomalies (on land and at sea) and geoid heights (GPS/leveling on land and GEOSAT-GM altimetry data at sea). The noise level is 10 cm for GPS/leveling and GEOSAT-GM data and 3 mGal for land and shipborne gravity data. Due to the lack of specific information about the errors in both altimetric and gravimetric solutions, simulated noises were used as input error. Randomly distributed fields (white noise) were generated in Matlab® using 10 cm standard deviation for the altimetry derived geoid heights and 3 mGal standard deviation for the gravimetric one. The final solutions from the combination method were calculated according to the following equation:

$$\hat{X}_0 = \hat{H}_{x_0 y_0} Y_0 = H_{xy}(P_{y_0 y_0} - P_{mm}) P^{-1}_{y_0 y_0} Y_0 \tag{3}$$

where $\hat{X}_o$ is estimated output spectrum, $\hat{H}_{x_0 y_0}$ is optimal frequency impulse response, Y_0 is the input observation spectrum, $P_{y_0 y_0}$ is the input observation PSD, P_{mm} is the input error PSD, and H_{xy} is the theoretical frequency impulse response of the system.

Solution 3: The most recent gravimetric geoid of Canada, CGG2000. The CGG2000 residuals to EGM96 show a mean value of –0.680 m with a standard deviation of 0.268 m.

5 Comparison and Validation of the MIMOST Solution

Before the comparison of our numerical solution to the MIMOST solution, it is necessary to validate the MIMOST solution itself. The solution using the MIMOST method is compared with GPS/leveling and GEOSAT GM data, the 2D FFT spherical Stokes solution with gravity anomalies, and the CGG2000 geoid. GPS and GEOSAT data are referenced to EGM 96.

Table 1. Statistics of MIMOST solution. Unit: [m].

Solution (data)	max	min	mean	std
MIMOST	0.759	-0.592	0.191	±0.154

Table 2. Statistics of differences of MIMOST solution from other solutions. Unit: [m].

Differences with	max	min	mean	std
GPS&GEOSAT	0.970	-1.071	-0.003	±0.209
Solution 1	1.264	-0.473	0.366	±0.209
CGG2000	1.740	-0.194	0.926	±0.267

The MIMOST solution is presented graphically in Figure 1. The smoother surface on land is due to the influence of the resolution of GPS/leveling data. The MIMOST solution is closest to the data from GPS/levelling and GEOSAT (see table 2) because of the higher apriory accuracy used for GPS and GEOSAT data. Differences with Solution 1 (see table 2) are due to the small weight of the gravity anomalies in MIMOST. The differences between the mean values in table 2 are due to the different referencing. CGG2000 refers to the mean sea level at several tide gauges; altimetry data are related to the sea surface and solution 1 is referenced to the geoid (sea surface corrected for the QSST).

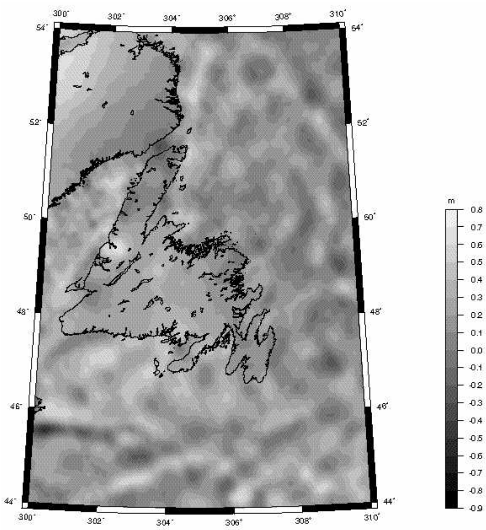

Fig. 1: MIMOST solution (residuals to EGM96).

6 Comparison and Validation of the Numerical Solution of AGBVP II

To validate the numerical solution of AGBVP II, it is compared to the 2D FFT spherical Stokes solution with gravity anomalies, the CGG 2000 geoid and the MIMOST solution.

Table 3. Statistics of numerical solution. Unit: [m].

Solution (data)	max	min	mean	std
Num. solution	0.418	-0.821	-0.241	±0.151

Table 4. Statistics of differences of the numerical solution from other solutions. Unit: [m].

Differences with	max	min	mean	std
Solution 1	0.349	-0.371	-0.058	±0.042
CGG2000	1.476	-0.539	0.459	±0.310
MIMOST	0.398	-1.236	-0.419	±0.205

The numerical solution is closest to Solution 1. The differences with Stokes's solution are negligible. The differences are due to GPS/leveling and GEOSAT data. In the numerical solution GPS/leveling and GEOSAT (corrected for the QSST) data can be considered as part of known boundary. The solution depends mostly on gravity data. Differences with CGG2000 are mainly due to the QSST which was neglected in the CGG2000 solution; the numerical solution does not contain the effect of QSST. After restoring the effect of QSST on the numerical solution, the mean value of the differences with CGG2000 is close to zero.

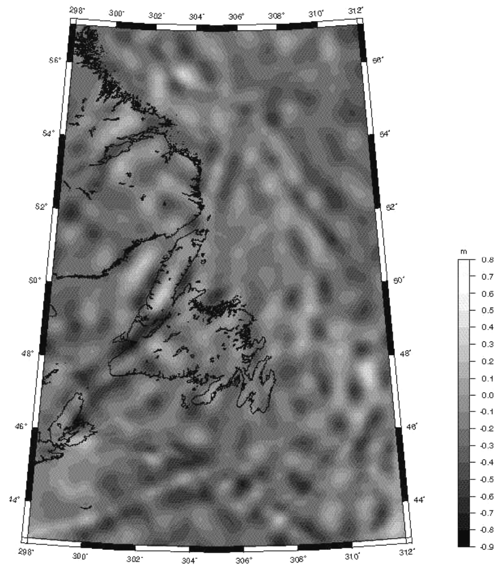

Fig. 2: Numerical solution (residuals to EGM96).

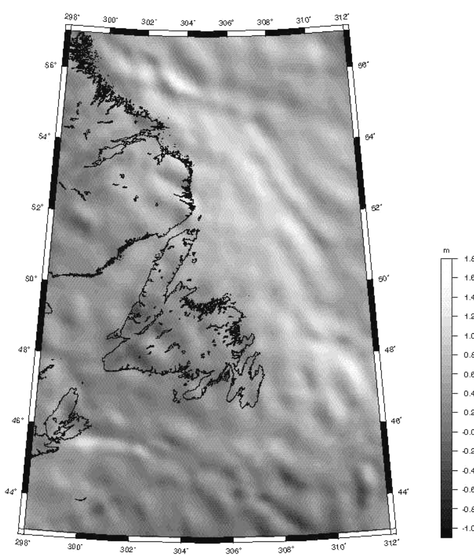

Fig. 3: Differences between numerical solution and CGG2000.

The main contribution to the numerical solution comes from gravity data. In terms of mean value, the numerical solution is closer to CGG2000, which shows again the gravimetric character of the numerical solution. A comparison with the MIMOST solution shows a better geoid modeling on land (see Figure 1 and Figure 2),

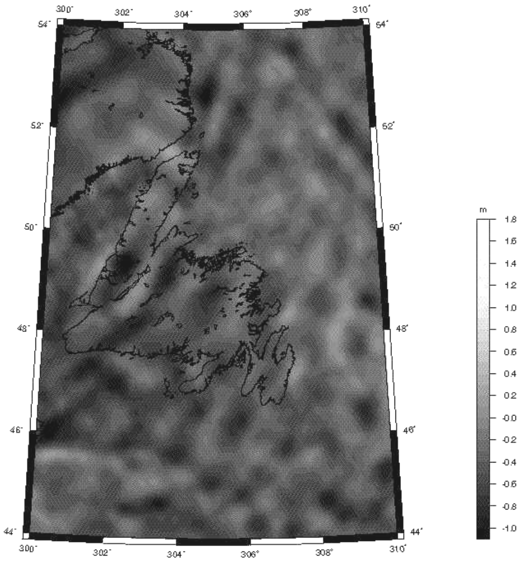

Fig. 4: Differences between numerical solution and MIMOST.

which can be seen in the differences between numerical and MIMOST solutions (Figure 3), as well.

Both the numerical solution and the MIMOST solution have been compared to a previous altimetry and shipborne gravity solution at sea only (Vergos et al., 2001). The changes in the coastline region due to the gravity data and GPS/leveling on land exist in both solutions. At the same time, a comparison to the solution with altimetry and shipborne data only at sea shows that the numerical solution is closer in terms of mean value (see table 5) and the standard deviations are very close.

Table 5. Statistics of differences to the altimetry and shipborne solution at sea only. Unit: [m].

Differences	max	min	mean	std
Num. Solution	0.645	-0.892	0.005	±0.240
MIMOST	1.257	-0.628	0.424	±0.232

7 Effect of Smoothing Conditions along the Coastline on the Numerical and MIMOST Solutions

In numerical and MIMOST solutions gravity data are considered as boundary conditions and at the same time, GPS/leveling and GEOSAT could be considered as part of known boundary surface. Smoothing conditions along the coastline are applied on boundary surface and boundary conditions (data) before the computation of the geoid. The effects of smoothing along the coastline are the differences between the geoid determined from original data and the geoid determined from smoothed data along the coastline. For the numerical solution, the effects of smoothing conditions on the boundary surface and on the data have been investigated separately. The gray scale in Figure 6 is different from the values in the table 6, because the figure represents a smaller area.

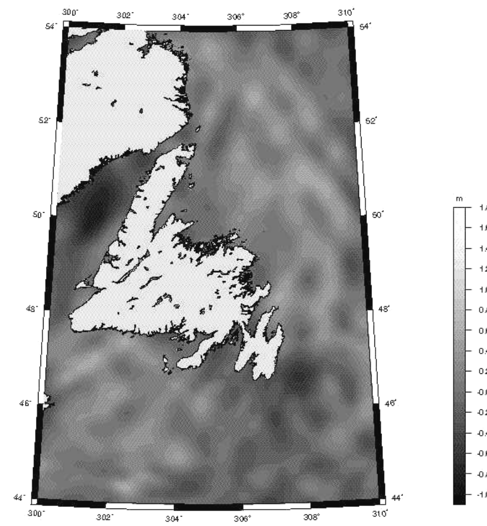

Fig.5: Differences between numerical solution and altimetry and shipborne solution at sea only

Table 6. Statistics of smoothing effects. Unit: [m].

Differences	max	min	mean	std
Num. Solution	0.645	-0.892	0.005	±0.240
Num. Solution-boundary	0.001	-0.001	0.000	±0.001
MIMOST	1.257	-0.628	0.424	±0.232

The smoothing effects (see table 6) on the numerical solution range between -0.137m and 0.103 m. For MIMOST, they have grater magnitude, between -0.234 m and 0.165 m. For the MIMOST method, the smoothing effects on the boundary (GPS and altimetry data) are larger, because of larger weight of these data in the combined solution and existing discrepancies between GPS and altimetry data. The smoothing of the boundary surface does not have an effect on the geoid (see table 6), in the numerical solution. The MIMOST solution is more sensitive to the smoothing on the boundary surface than the numerical solution. In both cases, the smoothing effects are concentrated in the coastal region.

8 Conclusions

After the analysis of the results obtained, the following conclusions are drawn:

- The suggested numerical solution with gravity disturbances as boundary data both on land and at sea can be successfully applied for the solution of the fixed AGBVP II.

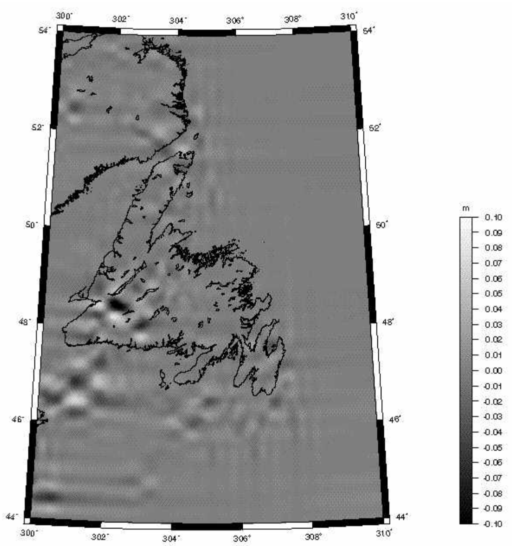

Fig. 6: Effect of smoothing along the coastline on the numerical solution

- The numerical solution is closer to the pure gravity solution taking into account GPS and GEOSAT data. These data describe the boundary surface and take part in the solution implicitly through the transformation of the boundary condition.
- The significant differences with CGG2000 are because the numerical solution does not contain the effect of QSST while CGG2000 does. After restoring the effect of SST in numerical solution, the mean value of the differences became zero.
- The smoothing of the boundary surface does not have an effect on geoid determination – the magnitude of this effect is 1mm!
- The smoothing on the boundary conditions (data) gives a maximum effect on the final geoid solution between -0.137 m and 0.103 m.
- The numerical solution is less sensitive to discrepancies between GPS/leveling and altimetry data. Even greater discrepancies between GPS/levelling and altimetry data do not have smoothing effects along the coastline.

Acknowledgements

The project is supported by grants from the GEOIDE Network of Centres of Exellence and NSERC.

References

Andritsanos V D, Sideris M G, Tziavos I N (2000) A survey of gravity field modelling applications of the Input-Output System Theory (IOST). IGeS Bulletin, vol. 10, 1-17.

Arnold K (1983) The mixed boundary value problem of geodesy, Gerlands Beitr. Geophys. 92, 391-406.

Grebenitcharsky R, Sideris M G (2001a) Altimetry-gravimetry boundary value problems with smoothness conditions in coastal regions, XXVI General Assembly of EGS, Nice, France, 25-30, March, 2001.

Grebenitcharsky R, Sideris M G (2001b) A comparison of different dolution methods for altimetry-gravimetry boundary value problems using smoothing conditions along the coastline, A poster presented in the 2001 IAG Scientific Assembly, Budapest, Hungary, September 2-7.

Holota P (1980) The Altimetry-gravimetry boundary value problem, presented at the International Scientific Conference of Sec. 6 Intercosmos, Albena.

Holota P (1983a) The altimetry gravimetry boundary value problem I: linearization, Friedrich's inequality. Boll. Geod. Sci. Afini., 42, 14-32.

Holota P (1983b) The altimetry gravimetry boundary value problem II: weak solution, V-ellipticity. Boll Geod. Sci. Afini., 42, 70-84.

Holota P (1997) Variational methods for geodetic boundary value problems In: Sanso F., Rummel R. (ed) Geodetic boundary value problems in view of the one centimeter geoid. Lecture Notes in Earth Sciences, vol. 65, Springer, Berlin Heidelberg New York, 469-481.

Lehmann R (1999) Boundary value problems in the complex world of geodetic measurements, J of Geodesy, vol. 73, 491-500.

Mallat S (1998) A wavelet tour of signal processing Academic Press, San Diego.

National oceanographic and atmospheric administration– NOAA (1997) The GEOSAT-GM Altimeter JGM-3 GDRs.

Rektorys K (1977) Variational methods in mathematics, science and engineering, D. Reidel Publishing Company, Dordecht/ Boston.

Sanso F (1981) Recent Advances in the theory of the geodetic boundary value problem, Review of Geoph. and Space Physics, vol. 19, No 3, 437-449.

Sanso F (1993) Theory of geodetic boundary value problems applied to the analysis of altimeter data, in: Rummel, R. and Sanso, F. (eds.) Satellite altimetry in geodesy and oceanography, Lecture Notes in Earth Sciences, Vol. 50, 318-371.

Sideris M G (1996) On the use of heterogeneous noisy data in spectral gravity field modeling methods, J of Geodesy, vol. 70, 470-479.

Svensson L (1983) Pseudodifferential operators – New Approach to the boundary problems of physical geodesy, Manuscr Geod, 8, 1-40.

Svensson L (1988) Some remarks on the altimetry-gravimetry problem. Manuscr Geod, 13, 63-74.

Vergos G, Grebenitcharsky R, Sideris M G (2001) Combination of multi-satellite altimetry and shipborne gravity data for geoid determination in a coastal region of eastern Canada, XXVI General Assembly of EGS, Nice, France, 25-30 March, 2001.